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Exact solutions for some coupled nonlinear equations: II

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Abstract. This work is the continuation of paper I [1]. Here we use the method in [1] to obtain the exact solutions for some coupled nonlinear equations.

As we know, coupled nonlinear equations will arise when we consider more than one type of interaction and more than one type of component in real physical systems [2-4]. To find the exact solution of the coupled nonlinear equations will be quite important in order to obtain a knowledge of the system. Here we deal with three kinds of coupled nonlinear equations by the method in [1].

The first coupled equations we deal with are the coupled nonlinear Schrödinger equations that arise in the study of monomode step-index optical fibres [2]

$$\mathrm{i}\frac{\partial A^+}{\partial \tau}=\frac{\partial^2 A^+}{\partial x^2}+A^+(|A^+|^2+h|A^-|^2) \tag{1}$$

$$\mathrm{i}\frac{\partial A^-}{\partial \tau}=\frac{\partial^2 A^-}{\partial x^2}+A^-(|A^-|^2+h|A^+|^2) \tag{2}$$

where A^+ and A^- are the amplitudes of the electric field and h is a parameter (see [2]).

In [2], Newboult *et al* have obtained some exact solutions of the coupled equations (1) and (2). Here we present another type of exact solution of it. Let

$$A^+=A_1(x)\exp(\mathrm{i}\rho_1\tau) \tag{3}$$

$$A^-=A_2(x)\exp(\mathrm{i}\rho_2\tau) \tag{4}$$

where ρ_1 and ρ_2 are two parameters.

The equations (1) and (2) will become

$$\frac{\partial^2 A_1}{\partial x^2}+\rho_1 A_1+A_1(|A_1|^2+h|A_2|^2)=0 \tag{5}$$

$$\frac{\partial^2 A_2}{\partial x^2}+\rho_2 A_2+A_2(|A_2|^2+h|A_1|^2)=0. \tag{6}$$

We make an ansatz for the solution:

$$A_1(x)=a\tanh\mu x \tag{7}$$

$$A_2(x)=b\,\mathrm{sech}\,\mu x \tag{8}$$

where a, b and μ are the parameters to be determined.

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Inserting equations (7) and (8) into equations (5) and (6), equating the same power of tanh μx and sech μx, respectively, we obtain the following parametric equations:

$$-2a\mu^2+\rho_1 a+hab^2=0 \tag{9}$$

$$2a\mu^2+a^3-hab^2=0 \tag{10}$$

$$b\mu^2+\rho_2 b+hba^2=0 \tag{11}$$

$$-2b\mu^2+b^3-hba^2=0. \tag{12}$$

From equations (9) to (12), we have

$$a=\pm\sqrt{-\rho_1} \tag{13}$$

$$b=\pm[(2\rho_2-\rho_1)/(h-2)]^{1/2} \tag{14}$$

$$\mu=[\rho_1(1+h)/2]^{1/2} \tag{15}$$

and one constraint equation for ρ_1 and ρ_2:

$$\rho_1=2\rho_2/(h-1). \tag{16}$$

Thus we obtain one type of exact solution with one arbitrary parameter ρ_1 (or ρ_2) as follows:

$$A^+=\pm\sqrt{-\rho_1}\tanh\left[\pm\left(\frac{h+1}{2}\rho_1\right)^{1/2}x\right]\exp\,(\mathrm{i}\rho_1\tau) \tag{17}$$

$$A^-=\pm\sqrt{\rho_1}\,\mathrm{sech}\left[\pm\left(\frac{h+1}{2}\rho_1\right)^{1/2}x\right]\exp\left(\mathrm{i}\frac{\rho_1(h+1)}{2}\tau\right). \tag{18}$$

As the coupled equations (5) and (6) have some symmetry for the variables A^+ and A^-, we also have the following solutions:

$$A^+=\pm\left(\frac{2\rho_1}{h-1}\right)^{1/2}\mathrm{sech}\left[\pm\left(\frac{h+1}{h-1}\rho_1\right)^{1/2}x\right]\exp(\mathrm{i}\rho_1\tau) \tag{17$'$}$$

$$A^-=\pm\left(\frac{2\rho_1}{1-h}\right)^{1/2}\tanh\left[\pm\left(\frac{h+1}{h-1}\rho_1\right)^{1/2}x\right]\exp\left(\mathrm{i}\frac{2\rho_1\tau}{h-1}\right). \tag{18$'$}$$

Following the same formulae of [2], we obtain the electric field as follows:

$$\begin{aligned}E^{(1)}=&\pm2\sqrt{-\rho_1}\tanh\left[\pm\left(\frac{h+1}{2}\rho_1\right)^{1/2}\left(\frac{f_2}{g}\right)^{1/2}\nu(z-st)\right]\\&\times\left\{-\tilde{E}_1\boldsymbol{e}_r\sin\left[\theta+\left(k+\frac{f_2\rho_1}{f_1}\nu^2\right)z-\omega t\right]\right.\\&\left.+(\tilde{E}_2\boldsymbol{e}_\theta+\tilde{E}_3\boldsymbol{e}_z)\cos\left[\theta+\left(k+\frac{f_2\rho_1\nu^2}{f_1}\right)z-\omega t\right]\right\}+2(\pm\sqrt{\rho_1})\\&\times\mathrm{sech}\left[\left(\frac{h+1}{2}\rho_1\right)^{1/2}\left(\frac{f_2}{g}\right)^{1/2}\nu(z-st)\right]\\&\times\left\{-\tilde{E}_1\boldsymbol{e}_r\sin\left[-\theta+\left(k+\frac{(h-1)\rho_1f_2\nu^2}{2f_1}\right)z-\omega t\right]\right.\\&\left.+(-\tilde{E}_2\boldsymbol{e}_\theta+\tilde{E}_3\boldsymbol{e}_z)\cos\left[-\theta+\left(k+\frac{(h-1)\rho_1f_2\nu^2}{2f_1}\right)z-\omega t\right]\right\}\end{aligned} \tag{19a}$$

$$E^{(1)}=\pm 2\left(\frac{2\rho_1}{h-1}\right)^{1/2}\operatorname{sech}\left[\left(\frac{h+1}{h-1}\rho_1\right)^{1/2}\left(\frac{f_2}{g}\right)^{1/2}\nu(z-st)\right]$$
$$\times\left\{-\tilde{E}_1\boldsymbol{e}_r\sin\left[\theta+\left(\frac{f_2\rho_1\nu^2}{f_1}+k\right)z-\omega t\right]\right.$$
$$\left.+(\tilde{E}_2\boldsymbol{e}_\theta+\tilde{E}_3\boldsymbol{e}_z)\cos\left[\theta+\left(k+\frac{f_2\rho_1\nu^2}{f_1}\right)z-\omega t\right]\right\}$$
$$+2\left[\pm\left(\frac{2\rho_1}{1-h}\right)\right]\tanh\left[\pm\left(\frac{h+1}{h-1}\rho_1\right)^{1/2}\left(\frac{f_2}{g}\right)^{1/2}\nu(z-st)\right]$$
$$\times\left\{-\tilde{E}_1\boldsymbol{e}_r\sin\left[-\theta-\omega t+\left(k+\frac{2\rho_1 f_2\nu^2}{(h-1)f_1}\right)z\right]\right.$$
$$\left.+(-\tilde{E}_2\boldsymbol{e}_\theta+\tilde{E}_3\boldsymbol{e}_z)\cos\left[-\theta+\left(k+\frac{2\rho_1 f_2\nu^2}{(h-1)f_1}\right)z-\omega t\right]\right\}. \tag{19b}$$

The second coupled nonlinear equation is the extension of the coupled nonlinear equations in [3], which can be written as follows:

$$U_t+\alpha V^2 V_x+\beta U^2 U_x+\lambda U U_x+\gamma U_{xxx}=0 \tag{20}$$

$$V_t+\delta(UV)_x+\varepsilon_0 V V_x+\varepsilon_1 V_{xx}+\varepsilon_2 V_{xxx}=0 \tag{21}$$

where α, β, λ, γ, δ, ε_0, ε_1 and ε_3 are parameters.

When $\varepsilon_1=\varepsilon_2=0$, equations (20) and (21) reduce to the case that was treated in [3].

We look for travelling solutions of equations (20) and (21), that is, we assume that

$$U(x,t)=U(x-\omega t)\equiv U(\xi) \tag{22}$$

$$V(x,t)=V(x-\omega t)\equiv V(\xi). \tag{23}$$

Inserting equations (22) and (23) into equations (20) and (21), and integrating them, we get

$$-\omega U+\tfrac{1}{3}\alpha V^3+\tfrac{1}{3}\beta U^3+\tfrac{1}{2}\lambda u^2+\gamma U_{\xi\xi}+C_0=0 \tag{24}$$

$$C_1-\omega V+\delta U V+\tfrac{1}{2}\varepsilon_0 V^2+\varepsilon_1 V_\xi+\varepsilon_2 V_{\xi\xi}=0 \tag{25}$$

where C_0 and C_1 are two integration constants.

In [3], they take C_0 to be zero, in order to obtain the exact solution by integrating equations (24) and (25). Here we show that when $C_0\neq 0$, we also have a similar exact solution.

In the following, we treat cases with different parameters.

Case A. $\varepsilon_1=\varepsilon_2=0$. We assume

$$U=a+b\tanh u\xi \tag{26}$$

$$V=c+d\tanh\mu\xi. \tag{27}$$

Inserting equations (26) and (27) into equations (24) and (25), equating the same power of tanh $\mu\xi$, we get the following parametric equations:

$$\tfrac{1}{2}\varepsilon c^2+\delta ac+C_1-\omega c=0 \tag{28}$$

$$cd\varepsilon_0+\delta(ad+bc)-\omega d=0 \tag{29}$$

$$\tfrac{1}{2}\varepsilon_0 d^2+\delta bd=0 \tag{30}$$

$$-\omega a+\tfrac{1}{3}\alpha c^3+\tfrac{1}{3}\beta a^3+\tfrac{1}{2}\lambda a^2+C_0=0 \tag{31}$$

$$-\omega b+c^2 d\alpha+a^2 b\beta+\lambda ab-2b\mu^2\gamma=0 \tag{32}$$

$$cd^2\alpha+ab^2\beta+\tfrac{1}{2}\lambda b^2=0 \tag{33}$$

$$\alpha d^3+\beta b^3+6b\gamma\mu^2=0. \tag{34}$$

From equations (28) to (24), we get

$$c=(\lambda\delta\varepsilon_0^2+2\beta\omega\varepsilon_0^2)/(\varepsilon_0^3\beta-8\delta^3\alpha) \tag{35}$$

$$a=(2\omega-\varepsilon_0 c)/2\delta \tag{36}$$

$$\mu=\left[\frac{1}{2\gamma}\left(\lambda a+a^2\beta-\omega-\frac{2\delta c^2\alpha}{\varepsilon_0}\right)\right]^{1/2} \tag{37}$$

$$b=\left(\frac{8\alpha\delta^3}{6\gamma\varepsilon_0^3}-\frac{\beta}{6\gamma}\right)^{-1/2}\mu \tag{38}$$

$$d=-(2\delta/\varepsilon_0)b. \tag{39}$$

When we do the following parametric transform in equations (37) to (39) respectively:

$$\frac{1}{2\gamma}\rightarrow-\frac{1}{\gamma} \tag{40}$$

$$\frac{1}{6\gamma}\rightarrow-\frac{1}{6\gamma} \tag{41}$$

equations (24) and (25) have the following solutions:

$$U=a+b\operatorname{sech}\mu\xi \tag{42}$$

$$V=c+d\operatorname{sech}\mu\xi \tag{43}$$

where the parameters a, b, c, d and μ are the same as those for the solution of equations (26) and (27), but taking the transform of equations (40) and (41) into consideration in equations (37) and (39).

The solution of equation (26) and equation (27) is similar to that in reference (3), but this time the integration constant is not equal to 0. The solution of equation (42) and equation (43) are new and cannot be obtained by the method in reference (2).

Case B. $\varepsilon_2=0$. This case cannot be treated by the method used in [3], as the two variables cannot reduce to one variable. However, following the same procedure as above, we obtain the exact solution in the form of a simple combination of hyperbolic functions. For convenience we only give the results.

We have

$$U = a + b \tanh \mu\xi \tag{44}$$

$$V = c + d \tanh \mu\xi \tag{45}$$

where

$$\mu = \pm\left(\frac{1}{2\gamma}\right)^{1/2} [(\lambda - \delta)a + a^2\beta - c\varepsilon_0 \pm c^2\alpha A_0^{-1} \pm \delta c A_0]^{1/2} \tag{46}$$

$$\omega = a\delta + c\varepsilon_0 \mp \delta c A_0 \tag{47}$$

$$b = B_0/C_0 \tag{48}$$

$$d = 2(\varepsilon_1\mu - \delta b)/\varepsilon_0 \tag{49}$$

with

$$A_0 = \left(\frac{-c\alpha}{\lambda/2 + a\beta}\right)^{1/2} \tag{50}$$

$$B_0 = \varepsilon_1\mu(a\delta + \varepsilon_0 c - \omega) \tag{51}$$

$$C_0 = \delta(a\delta + \tfrac{1}{2}c\varepsilon_0 - \omega) \tag{52}$$

and one constraint equation for the parameters a, c, λ, γ, α, β, ε_0 and ε_1:

$$\alpha d^3 + \beta b^3 + 6\gamma b\mu^{2} * 0. \tag{53}$$

Case C. $\varepsilon_1 = 0$. We have

$$U = a + c \tanh^2 \mu\xi \tag{54}$$

$$V = d + f \tanh^2 \mu \tag{55}$$

where

$$c = -\left(\frac{\alpha}{\beta}\right)^{1/3} f \equiv \alpha_0 f$$

$$\mu^2 = \frac{1}{6\varepsilon_2}\left(\delta\alpha_0 + \frac{\varepsilon_0}{2}\right) f \equiv Ff \tag{56}$$

$$a = \delta[8\mu^2\varepsilon_2 + \omega - d(\varepsilon_0 + \delta\alpha_0)] \equiv \frac{8\varepsilon_2}{\delta} Ff + \frac{\omega}{\delta} + Ed \tag{57}$$

$$d = \left[\frac{\gamma\delta\alpha_0^2}{\varepsilon_2} + \frac{\gamma\alpha_0\varepsilon_0}{2\varepsilon_2} - \frac{\lambda\alpha_0^2}{2} - \frac{\beta\alpha_0^2\omega}{\delta} + \frac{4\beta\alpha_0^2}{3\delta} \times \left(\delta\alpha_0 + \frac{\varepsilon_0}{2}\right) f\right]\left(\alpha - \frac{\beta\alpha_0^2(\delta\alpha_0 + \varepsilon_0)}{\delta}\right)^{-1} \tag{58}$$

$$\equiv A + Bf \tag{59}$$

and one parameter constraint equation relating f, α, β, λ, γ, δ, ε_0 and ε_2:

$$f^2\left[\alpha B^2+\beta\alpha_0\left(EB+\frac{8\varepsilon_2}{\alpha}F\right)^2\right]+f\left[2\alpha AB+2\left(EA+\frac{\omega}{\delta}\right)\left(EB+\frac{8\varepsilon_2}{\delta}F\right)\right.$$
$$\left.+\lambda\alpha_0\left(EB+\frac{8\varepsilon_2}{\delta}F\right)-8\alpha_0\gamma F\right]-\omega\alpha_0+\alpha A^2$$
$$+\beta\alpha_0\left(EA+\frac{\omega}{\delta}\right)^2+\lambda\alpha_0\left(EA+\frac{\omega}{\delta}\right)=0. \tag{60}$$

Case D. $\gamma=0$. We have

$$U=a+b\tanh\mu\xi+c\tanh^2\mu\xi \tag{61}$$

$$V=d+e\tanh\mu\xi+f\tanh^2\mu\xi \tag{62}$$

where

$$f=(-\beta/\alpha)^{1/3}c \tag{63}$$

$$\mu^2=\frac{\varepsilon_0}{12\varepsilon_2}\left[\left(\frac{\beta}{\alpha}\right)^{1/3}-\frac{2\delta}{\varepsilon_0}\right]c \tag{64}$$

$$e=2f\mu\varepsilon_1\left[\delta c+\varepsilon_0 f+2\mu^2\varepsilon_2-\delta\left(\frac{\alpha}{\beta}\right)^{1/3}f\right]^{-1} \tag{65}$$

$$b=-(\alpha/\beta)^{1/3}e \tag{66}$$

$$d=[\beta c(\delta c+\varepsilon_0 f)-\delta f\alpha c(\beta/\alpha)^{2/3}]^{-1}$$
$$\times[\beta c\omega f+\beta c\varepsilon_1\mu e+\delta f\mu^2\varepsilon_2\beta c-\delta be\beta c-\tfrac{1}{2}\beta c\varepsilon_0 e^2+\tfrac{1}{2}\delta fc\lambda] \tag{67}$$

$$a=(1/\delta f)[\omega f+\varepsilon_1\mu e+\delta f\mu^2\varepsilon_2-\delta be-(\delta c+\varepsilon_0 f)d-\tfrac{1}{2}\varepsilon_0 e^2] \tag{68}$$

where ω satisfies

$$[\beta B_1^2C_1^2-\alpha(\beta/\alpha)^{1/3}C_1^2]\omega^2+\omega[2B_1C_1\beta(B_1D_1+A_1)-1+\lambda B_1C_1$$
$$-\alpha(\beta/\alpha)^{1/3}2C_1D_1]+\beta(B_1D_1+A_1)^2$$
$$+\lambda(B_1D_1+A_1)\alpha(\beta/\alpha)^{1/3}D_1^2=0 \tag{69}$$

with

$$A_1=-\lambda/2\beta \tag{70}$$

$$B_1=-(\alpha/\beta)^{1/3} \tag{71}$$

$$C_1=[\beta c(\delta c+\varepsilon_0 f)-\delta f\alpha c(\beta/\alpha)^{2/3}]^{-1}\beta cf \tag{72}$$

$$D_1=[\beta c(\delta c+\varepsilon_0 f)-\delta f\alpha c(\beta/\alpha)^{2/3}]^{-1}$$
$$\times(\beta c\varepsilon_1\mu e+8f\mu^2\varepsilon_2\beta c-\delta be\beta c-\beta c\varepsilon_0 e^2+\tfrac{1}{2}\delta fc\lambda) \tag{73}$$

and two parameter constraint equations for c, α, β, λ, δ, ε_0, ε_1 and ε_2:

$$-\omega e+\delta(ae+bd)+\varepsilon_0 de+2f\varepsilon_1\mu-2e\mu^2\varepsilon_2=0 \tag{74}$$

$$-\omega c+[\alpha(de^2+d^2f)+\beta(ab^2+a^2c)]+\tfrac{1}{2}\lambda(b^2+2ac)-8c\mu^2\gamma=0. \tag{75}$$

The third coupled nonlinear equations are coupled equations, which can be written as follows (see [4]):

$$m(C_0^2 - V^2)\frac{d^2U}{ds^2} = AU - 2K\rho^2 U - BU^2 + CU^3 - DVU_s \tag{76}$$

$$M(V_0^2 - V^2)\frac{d^2\rho}{ds^2} = -M\Omega_0^2\rho - 2KP(U^2 - U_0^2) - EV\rho_s. \tag{77}$$

When the coupled interaction term has the form $K\rho(U^2 - U_0^2)$, and D and E are zero, the coupled equations (76) and (77) reduce to the case that was considered in [4].

We discuss the different parameter cases separately.

Case (i). All parameters not equal to zero. We have two types of solutions.

The first type of solution can be written as follows:

$$U = \alpha + \beta \tanh \mu s \tag{78}$$

$$\rho = \gamma + \delta \tanh \mu s \tag{79}$$

where

$$\beta = \left(\frac{M(V^2 - V_0^2)}{K}\right)^{1/2} \mu \equiv \beta_0\mu \tag{80}$$

$$\delta = \left(\frac{C\beta_0^2 - 2m(C_0^2 - V^2)}{2K}\right)^{1/2} \mu \equiv \delta_0\mu \tag{81}$$

$$\alpha = \frac{2EV\delta_0^2 - DV\beta_0^2 + B\beta_0^3}{3\beta_0^3 C + 6K\beta_0\delta_0^2} \tag{82}$$

$$\gamma = (EV\delta_0 - 4K\alpha\delta_0\beta_0)/2K\beta_0^2 \tag{83}$$

$$\mu = \pm[(A\alpha - 2K\alpha\gamma^2 - B\alpha^2 + c\alpha^3)/DV\beta_0]^{1/2} \tag{84}$$

and three parameter constraint equations for A, B, C, D, E and V:

$$\delta_A E\alpha = (2K\alpha\gamma^2 + B\alpha^2 - C\alpha^3)E\delta_0 - D[m\Omega_0^2\gamma + 2K\gamma(U_0^2 - \alpha^2)]\beta_0 \tag{85}$$

$$-2\beta_0\mu^2 m(C_0^2 - V^2) = A\beta_0 - 2K(2\gamma\delta_0\alpha + \beta_0\gamma^2) - 2B\alpha\beta_0 + 3\alpha^2\beta_0 C \tag{86}$$

$$2\delta_0\mu^2 M(V_0^2 - V^2) = M\delta_0\Omega_0^2 + 2K[2\alpha\beta_0\gamma + \delta_0(\alpha^2 - U_0^2)]. \tag{87}$$

The second type of solution is as follows:

$$U = \alpha + \beta \tanh \mu\xi \tag{88}$$

$$\rho = \delta \operatorname{sech} \mu\xi \tag{89}$$

where

$$\alpha = \pm EV/4[KM(V^2 - V_0^2)]^{1/2} \tag{90}$$

$$\beta = \pm[M(V^2 - V_0^2)/K]^{1/2}\mu \equiv \beta_0\mu \tag{91}$$

$$\delta = \pm\{[2m(C_0^2 - V^2) - C\beta_0]/2K\}^{1/2}\mu \equiv \delta_0\mu \tag{92}$$

$$\mu = \pm[(2B\alpha - A - 3C\alpha^2)/C\beta_0^2]^{1/2} \tag{93}$$

$$B = \frac{\alpha}{\beta_0^2}\left(2m(C_0^2 - V^2) + 2\beta_0^2 C + \frac{DV\beta_0}{\alpha}\right) \tag{94}$$

and D satisfies

$$A_3D^2+B_3D+C_3=0 \tag{95}$$

where

$$A_3=-2V^2\alpha^2/C^2\beta_0^2 \tag{96}$$

$$B_3=V(A+3C\alpha^2)/C\beta_0-\frac{2V\alpha^2D_3}{C\beta_0}-\frac{\alpha^2V}{C\beta_0^3}[4m(C_0^2-V^2)-C\beta_0^2] \tag{97}$$

$$C_3=A\alpha+C\alpha^3-\alpha^2D_3\frac{4m(C_0^2-V^2)-C\beta_0^2}{C\beta_0^2}+\frac{\alpha(A+3C\alpha^2)}{C\beta_0^2}[2m(C_0^2-V^2)-CB_0^2] \tag{98}$$

$$D_3=\frac{\alpha}{\beta_0^2}[2m(C_0^2-V^2)+2\beta_0^2C] \tag{99}$$

and one constraint equation for C and A:

$$-M\Omega_0^2=\mu^2M(V_0^2-V^2)+2K(\alpha^2+\beta^2-U_0^2). \tag{100}$$

Case (ii). B = 0. We shall have

$$U=\beta \operatorname{sech} \mu s \tag{101}$$

$$\rho=\gamma+\delta\tanh\mu s \tag{102}$$

where

$$\beta=\left(\frac{M(V_0^2-V^2)}{K}\right)^{1/2}\mu\equiv\beta_0\mu \tag{103}$$

$$\delta=\left(-\frac{D\beta_0^2}{2E}\right)^{1/2}\equiv\delta_0\mu \tag{104}$$

$$\gamma=-EV\delta_0/2K\beta_0^2 \tag{105}$$

$$\mu=\left(\frac{A-2K\gamma^2}{m(C_0^2-V^2)+2K\delta_0^2}\right)^{1/2} \tag{106}$$

$$C=[2K\delta_0^2-2m(C_0^2-V^2)]/\beta_0^2 \tag{107}$$

and one constraint equations for A, B, C and K:

$$M\Omega_0^2=2KU_0^2. \tag{108}$$

When $D=E=0$, we may point out that equations (76) and (77) have the following algebraic soliton solution:

$$U=\frac{1}{a+bs^2} \tag{109}$$

$$\rho=\alpha+\frac{\beta}{a+bs^2} \tag{110}$$

where

$$\alpha = \pm(A/2K)^{1/2} \tag{111}$$

$$\beta = \frac{-B + (B^2 - 8K\alpha^2 C)^{1/2}}{4K\alpha} \tag{112}$$

$$a = -3\beta/4\alpha \tag{113}$$

$$b = -\frac{(4K\alpha\beta + B)}{6m(C_0^2 - V^2)} \tag{114}$$

$$V = \left(\frac{(\beta^2 - C/2K)V_0^2 - m/MC_0^2}{\beta^2 - C/2K - m/M}\right)^{1/2} \tag{115}$$

and one constraint equation for the parameters A, B, C and K:

$$M\Omega_0^2 = 2KU_0^2. \tag{116}$$

One can see from the above that the solutions for the coupled nonlinear equations are richer than that for a single nonlinear equation. We shall apply these exact solutions to real systems like those treated in [2-4]. This is our future work.

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